How to fool the GHZ and W witnesses

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In spite of the fact that there are only two classes of three-qubit genuine entanglement: W and GHZ, we show that there are three-qubit genuinely entangled states which can not be detected neither by W nor by GHZ entanglement witnesses.

Keywords: entanglement witness, detection of entanglement, tripartite entanglement

1. Introduction

Entanglement is presumably the most non-classical feature of quantum mechanics. It is in the heart of Einstein-Podolsky-Rosen paradox [1]; of the so-called *which way* interferometers [2], in which the interference pattern is washed out by the entanglement of the interferometric particle with a "which way discriminator"; of teleportation of quantum states [3]; of some cryptographic protocols [4]; and of the most important quantum algorithms [5]. It is not difficult to *define* when a quantum state is entangled: for pure states they can only be (in bipartite case) factorizable, when can be written as $|\psi\rangle \otimes |\phi\rangle$, or entangled. The multipartite case is a little bit subtler, since the parts can be put together. For example, for three qubits (A, B, and C), the state given by a Bell state [6] $|\Psi^{\pm}\rangle = \{|01\rangle \pm |10\rangle\} / \sqrt{2}$ of the pair AB together with a pure state of C is said *biseparable*, in the sense that, with respect to the bipartition $AB - C$, it has no entanglement at all. One usually says that in such a state there is no *genuine* tripartite entanglement [7]. For mixed states, whenever ρ can be written as an ensemble of pure states which do not have some kind of entanglement, the mixed state is also said not to have such kind of entanglement. However, it is not a simple task to identify whether a given state has some kind of entanglement. Peres obtained a good criterion by noting that for separable states, the *partial transposition* lead to valid density matrices, whereas for some entangled state a non-positive matrix is generated [8]. The Horodeckis put this contribution

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in the context of *positive maps which fail to be completely positive* acting on density operators [9], and showed that the criterion is sufficient only for two qubits or for one qubit and one qutrit. Examples of bipartite entangled states which have positive partial transpose (PPT) are known [10].

Another way of identifying entanglement comes from the fact that, for a given type of entanglement, the set of (mixed) states which are free from such entanglement is a closed convex set. Any point in the complement of a closed convex subset of a vector space can be separated from it by a hyperplane [11]. Equivalently, there is a linear functional over this space which is positive in all separable states σ (*i.e.*: those which do not pursue the inspected kind of entanglement), and negative in the specific entangled state ρ (*i.e.*: the point out of the closed convex subset). This functional is called an *entanglement witness*, and is represented by the operator $\mathcal{W}$ [12]. In linear algebraic context, it reads: $\mathrm{Tr}(\mathcal{W}\rho) < 0$ and $\mathrm{Tr}(\mathcal{W}\sigma) \geq 0$, for all separable σ. Once again, it is simple *in theory*, but given a state which one wants to decide whether it shows some kind of entanglement or not, it is not easy to find the appropriated witness. A very important class of entanglement witnesses (EW) is the optimal entanglement witnesses (OEW). An OEW is the EW that best witnesses the entanglement of ρ, in the sense that it reaches the maximum value of $\mid \mathrm{Tr}(\mathcal{W}\rho) \mid$ [13, 14]. Unfortunately, finding an OEW is a difficult task because it involves a hard process of optimization [15]. However some improvements has been done in the sense of finding OEWs for certain kinds of states [16, 17].

For three qubits, Dür, Vidal, and Cirac showed that there are two different genuine tripartite entangled states, in the sense that, even in a statistical sense, they can not be locally converted one into another [18]. These states are the so called GHZ state, which appeared on the literature in Ref. [19], $|GHZ\rangle = \{|000\rangle + |111\rangle\}/\sqrt{2}$, and the W state [20], $|W\rangle = \{|001\rangle + |010\rangle + |100\rangle\}/\sqrt{3}$. Ishizaka and Plenio showed that even with PPT-operations (those which send PPT operators into PPT operators), GHZ and W remain inequivalent [21], however, stochastically there is a protocol to convert GHZ in W via PPT-operations with approximately 75% of success [22].

2. Fooling the GHZ/W-criterion

A systematic way to construct entanglement witnesses to pure states is looking for the optimal element of the set of EWs that have the specific form [23]

$$\mathcal{W}_\psi = \Lambda - |\psi\rangle \langle\psi| , \tag{1}$$

with $\Lambda \in \mathcal{R}$ and multiplication by the identity operator omitted in notation. By the condition that $\mathrm{Tr}(\mathrm{W}\sigma) \geq 0$ if σ is separable (denote $\mathcal{S}$ this set), we can see that

$$\Lambda = \max_{\sigma \in \mathcal{S}} \| \langle \sigma \mid \psi \rangle \|^2. \tag{2}$$

As an example, for the specific case of three qubits, the OEW for the GHZ-like states,

$$|GHZ(\phi)\rangle = \frac{(|000\rangle + e^{i\phi}|111\rangle)}{\sqrt{2}} \tag{3}$$

and for the W-like states,

$$|W(\gamma, \beta)\rangle = \frac{(|001\rangle + e^{i\gamma}|010\rangle + e^{i\beta}|100\rangle)}{\sqrt{3}}, \tag{4}$$

are

$$\mathcal{W}_{GHZ(\phi)} = \frac{1}{2} - |GHZ(\phi)\rangle \langle GHZ(\phi)| \tag{5}$$

and

$$\mathcal{W}_{W(\gamma,\beta)} = \frac{2}{3} - |W(\gamma,\beta)\rangle \langle W(\gamma,\beta)| \,, \tag{6}$$

respectively. Motivated by the existence of the two classes of genuine three qubit entanglement, some authors have used the GHZ-witness and the W-witness as a test of existence of genuine tripartite entanglement. We call this test the GHZ/W-criterion and, as we shall see, this criterion is incomplete in the sense that there are genuine entangled states which are not detected by it.

In the context of generalizing the Schmidt decomposition, Acín *et al.* showed that all three-qubit pure states can be written in the following way [24]:

$$|\psi\rangle = \lambda_0 |000\rangle + \lambda_1 e^{i\alpha} |001\rangle + \lambda_2 |010\rangle + \lambda_3 |100\rangle + \lambda_4 |111\rangle \,, \tag{7}$$

where $0 \le \lambda_i \in \mathcal{R}$, $0 \le \alpha \le \pi$, and $\sum_i \lambda_i^2 = 1$, by appropriate choices of local basis. Let us see what are the conditions for the GHZ(ϕ)-witness and the W(γ, β)-witness indicating entanglement in a general pure state.

GHZ(ϕ)-witness:

$$\mathrm{Tr}(\mathcal{W}_{\mathrm{GHZ}(\phi)} |\psi\rangle \langle\psi|) = \frac{1}{2} - \frac{1}{2}(\lambda_0^2 + \lambda_4^2 + 2\lambda_0\lambda_4 \cos(\phi - \alpha)) < 0. \tag{8}$$

As $\lambda_0, \lambda_4 \ge 0$, it is sufficient to our aim to consider $\phi = \alpha$. Thus,

$$(\lambda_0 + \lambda_4)^2 > 1, \tag{9}$$

must hold for the entanglement of $|\psi\rangle$ to be detected by GHZ(ϕ)-witness.

W(γ, β)-witness:

$$\begin{aligned} \mathrm{Tr}(\mathcal{W}_{\mathrm{W}(\gamma,\beta)} |\psi\rangle \langle\psi|) &= \frac{2}{3} - \frac{1}{3}\{\lambda_1^2 + \lambda_2^2 + \lambda_3^2 + 2\lambda_1\lambda_2 \cos(\gamma - \phi) + \\ &\quad 2\lambda_1\lambda_3 \cos(\beta - \phi) + 2\lambda_2\lambda_3 \cos(\gamma - \beta)\} < 0. \end{aligned} \tag{10}$$

Again, we consider the extremal case $\gamma = \phi = \beta$. Thus,

$$(\lambda_1 + \lambda_2 + \lambda_3)^2 > 2. \tag{11}$$

Within these conditions, it is easy to see that the state

$$|\xi\rangle = \frac{1}{\sqrt{5}}(|000\rangle + |001\rangle + |010\rangle + |100\rangle + |111\rangle) \tag{12}$$

is neither witnessed by W(γ, β)-witness nor by GHZ(ϕ)-witness. Note that $|\xi\rangle$ has genuine-tripartite entanglement, *i.e.*: it cannot be written as a biseparable state. In fact, $|\xi\rangle$ is a particular case of a whole family of pure states which are not witnessed by $\mathcal{W}_{GHZ(\phi)}$ and $\mathcal{W}_{W(\gamma,\beta)}$. The states

$$|\xi'\rangle = a\,|GHZ(\phi)\rangle + b\,|W(\gamma,\beta)\rangle \,, \tag{13}$$

with $|a|^2 + |b|^2 = 1$, reach

$$\langle \mathcal{W}_{GHZ(\phi)} \rangle = \frac{1}{2} - |a|^2, \tag{14}$$

and

$$\langle \mathcal{W}_{W(\gamma,\beta)} \rangle = \frac{2}{3} - |b|^2. \tag{15}$$

It means that $|\xi'\rangle$ is witnessed if $|a|^2 > \frac{1}{2}$ or $|b|^2 > \frac{2}{3}$. However these conditions exclude all states which satisfy $\frac{1}{3} \leq |a|^2 \leq \frac{1}{2}$ (see figure 1).

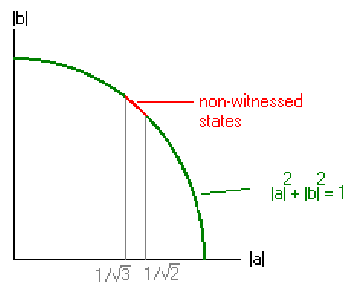

Fig. 1. Red: family of states $|\xi'\rangle = a\,|GHZ(\phi)\rangle + b\,|W(\gamma,\beta)\rangle$ which are not detected by the GHZ/W-criterion.

Furthermore statistic mixtures of unwitnessed pure states are also unwitnessed. To prove it we can take a look at the state:

$$\rho = \sum_i p_i \, |\xi'_i\rangle \langle \xi'_i| \, , \tag{16}$$

where the states $|\xi'_i\rangle$ are not witnessed by the GHZ/W-criterion. Applying the criterion to ρ we have:

$$\mathrm{Tr}(\mathcal{W}_{\mathrm{GHZ}(\phi)}\rho) = \sum_{\mathrm{i}} \mathrm{p_i} \mathrm{Tr}(\mathcal{W}_{\mathrm{GHZ}(\phi)} \, |\xi'_{\mathrm{i}}\rangle \langle \xi'_{\mathrm{i}}|) \geq 0 \tag{17}$$

and

$$\mathrm{Tr}(\mathcal{W}_{\mathrm{W}(\gamma,\beta)}\rho) = \sum_{\mathrm{i}} \mathrm{p_i} \mathrm{Tr}(\mathcal{W}_{\mathrm{W}(\gamma,\beta)} \, |\xi'_{\mathrm{i}}\rangle \langle \xi'_{\mathrm{i}}|) \geq 0, \tag{18}$$

where we have used the linearity of trace and that $p_i \geq 0$, $\mathrm{Tr}(\mathcal{W}_{\mathrm{GHZ}(\phi)} \, |\xi'_{\mathrm{i}}\rangle \langle \xi'_{\mathrm{i}}|) \geq 0$, and $\mathrm{Tr}(\mathcal{W}_{\mathrm{W}(\gamma,\beta)} \, |\xi'_{\mathrm{i}}\rangle \langle \xi'_{\mathrm{i}}|) \geq 0$. For sure, it is harder to guarantee which such combinations remain tripartite genuine entangled, however, continuity is enough to guarantee that some of them do remain.

3. Conclusion and Speculations

In conclusion, it was shown that testing the GHZ(ϕ)-witnesses and the W(γ, β)-witnesses is not a sufficient condition to attest if a general state is a genuine-entangled state. We have presented a family of pure states which fools the GHZ/W-criterion. For sure, this must not be a great surprise, since entanglement witness are taylored to indicate entanglement, not to deny it. We believe that our work can avoid mistakes on the detection of three-qubit entanglement in future works.

An interesting question raised by this incomplete criterion is to determine whether there is a finite set of witness operators that can determine if any state is a genuine-entangled state. Maybe it could be done through recent results on searching procedures for OEW [15, 25]. It can be viewed as a detection counterpart of the problem of determining the minimal set of entangled states that can generate all entangled states under certain operations (*e.g.*: LOCC) [21]. It could help, for instance, in understanding the geometry behind the set of separable states with respect to each kind of entanglement, in a complementary way to Ref. [26]. Another elucidating research would consist in finding what are the states witnessed by the class of witnesses reached through of different local basis of the GHZ and W states.

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